# PULSE PROPAGATION IN RANDOM MEDIA WITH LONG-RANGE CORRELATION

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**Abstract.** This paper analyses wave propagation in a one-dimensional random medium with long-range correlations. The asymptotic regime where the fluctuations of the medium parameters are small and the propagation distance is large is studied. In this regime pulse propagation is characterized by a random time shift described in terms of a fractional Brownian motion and a deterministic spreading described by a pseudo-differential operator. This operator is characterized by a frequency-dependent attenuation that obeys a power law with an exponent ranging from 1 to 2 that is related to the power decay rate of the autocorrelation function of the fluctuations of the medium parameters. This frequency-dependent attenuation is associated with a frequency-dependent phase, which ensures causality of the filter that realizes the approximation. A discussion is provided showing that the mean-field theory cannot capture the correct attenuation rate, this is because it also averages the random time delay. Numerical results are given to illustrate the accuracy of the asymptotic theory.

**Key words.** wave propagation, random media, long-range processes.

**1. Introduction.** Wave propagation in multiscale and rough media, with long-range fluctuations, has recently attracted a lot of attention, as more and more data collected in real environments confirm that this situation can be encountered in many different contexts, such as in geophysics [11] or in laser beam propagation through the atmosphere [13, 16, 25]. Recently it has been shown that the main effect of such fluctuations of the medium parameters is a random time shift for the wave front, that obeys a Gaussian statistics described in terms of a fractional Brownian motion [22]. Here we observe the wave front along its random characteristics and we show that the wave front also experiences a deterministic shape modification, that can be described in terms of a pseudo-differential operator that depends on the power decay rate of the autocorrelation function of the fluctuations of the medium parameters. These results extend to general long-range media the ones derived in the context of a discrete Goupillaud medium in [26].

The effective pseudo-differential operator obtained in this paper gives rise to a frequency-dependent attenuation that obeys a power law with an exponent ranging from 1 to 2. This exponent will be shown to be related to the exponent of the power decay rate of the autocorrelation function of the fluctuations of the medium parameters. Frequency-dependent attenuation has been observed in a wide range of applications in acoustics [4, 29], and also in other domains, such as seismic wave propagation [7, 8]. Experimental observations show that the attenuation of plane acoustic waves has a frequency dependence of the form $E = E_0 \exp(-\gamma(\omega) z)$, where $E$ denote the amplitude of an acoustic variable such as velocity or pressure and $\omega$ is the frequency. The damping coefficient has been seen to obey the empirical power law $\gamma(\omega) = \gamma_0 |\omega|^{\gamma_1}$ where $\gamma_0 \in (0, \infty)$ and $\gamma_1 \in (0, 2)$ are parameters characteristic of the medium obtained through a fitting of measured data. Different physical models exist that can predict such a power law [9, 17, 18, 28, 29]. One of the problems discussed in detail in these papers is to obtain a causal wave equation in the space-time domain that reproduces such a power law. In our paper we propose a derivation from first

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principles of an effective equation that exhibits a frequency-dependent attenuation with a power law, and we show that this attenuation is accompanied by a frequency-dependent phase that ensures the causality of the associated approximation.

Our approach is based on limit theorems and is valid when the fluctuations of the medium parameters are small and the propagation distance is large. We shall compare the results obtained with this asymptotic theory with the mean-field approach. We will show that, in the regime addressed in this paper, the mean-field approach gives a wrong prediction, because it averages out a random time shift that obeys Gaussian statistics, which gives rise to a non-physical diffusion. To describe the wave propagation phenomenon it is necessary to carry out a complete statistical analysis to identify the effective frequency-dependent attenuation.

The paper is organized as follows. We describe the acoustic wave model and the random medium in Section 2. We then state and discuss the most important results in Section 3, these results are then proved in Section 4. We present in Section 5 numerical simulations that illustrate the theoretical predictions of this paper. Finally, in Section 6, we briefly discuss the case with so-called anti-persistent scaling.

## 2. The Model.

### 2.1. Acoustic Wave Equations.

We develop an asymptotic probabilistic theory for the acoustic wave equations in the presence of random fluctuations of the medium with long-range correlation. The one-dimensional acoustic wave equations are given by

$$\rho(z)\frac{\partial u}{\partial t} + \frac{\partial p}{\partial z} = 0\,, \tag{2.1}$$

$$\frac{1}{K(z)}\frac{\partial p}{\partial t} + \frac{\partial u}{\partial z} = 0\,, \tag{2.2}$$

where $p$ is the pressure and $u$ is the velocity. For simplicity we assume that the density of the medium $\rho$ is a constant equal to $\overline{\rho}$. The bulk modulus of the medium $K$ is assumed to be randomly varying in the region $z \in [0, L]$ and we consider the weakly heterogeneous regime [15, 22], in which the fluctuations of the bulk modulus are small and rapid (compared to the propagation distance):

$$\frac{1}{K(z)} = \begin{cases} \frac{1}{\bar{K}}\left(1 + \varepsilon\nu(z/\varepsilon^2)\right) & \text{for } \ z \in [0, L]\,, \\ \frac{1}{\bar{K}} & \text{for } \ z \in (-\infty, 0) \cup (L, \infty)\,, \end{cases}$$

$$\rho(z) = \bar{\rho} \ \text{ for all } \ z\,.$$

The relevance of this long-range weakly heterogeneous regime is discussed in [22]. Here, the effective impedance and speed of sound are $\bar{\zeta} = \sqrt{\bar{K}\bar{\rho}}$ and $\bar{c} = \sqrt{\bar{K}/\bar{\rho}}$, respectively. The source located at $z_0 < 0$ emits a pulse at time $z_0/\bar{c}$. This pulse is impinging on the section $[0, L]$ and hits the boundary at 0 at time 0.

The random process $\nu$ is assumed to be bounded, stationary and with zero-mean. We assume in this paper that the medium has long-range correlation, in the sense that the autocorrelation function

$$\phi_0(z) := \mathbb{E}[\nu(y)\nu(y+z)] \tag{2.3}$$

is not integrable and has a power decay at infinity. Specifically, we assume

Assumption 1. *(i) $\nu$ is a bounded, stationary and zero-mean random process whose autocorrelation function satisfies:*

$$\phi_0(z) \overset{z\to\infty}{\sim} \frac{c_\alpha}{z^\alpha}\,, \tag{2.4}$$

*where $c_\alpha > 0$ and $\alpha \in (0,1)$. We call $l_c$ the critical length scale that corresponds to an inner scale below which the power law form behavior (2.4) is not valid.*

*(ii) $\nu$ satisfies the fourth-order moment conditions*

$$\mathbb{E}[\nu(y_1)\nu(y_2)\nu(z+y_3)\nu(z+y_4)] \overset{z\to\infty}{\longrightarrow} \mathbb{E}[\nu(y_1)\nu(y_2)]\mathbb{E}[\nu(y_3)\nu(y_4)] \tag{2.5}$$

*for all $y_1, y_2, y_3, y_4 \geq 0$,*

*(iii) $\nu$ is twice differentiable with bounded derivatives.*

The boundedness of $\nu$ is necessary to make the model physically relevant: the bulk modulus is a positive quantity so that $1+\varepsilon\nu$ must be positive. This holds for $\varepsilon$ small enough as soon as $\nu$ is bounded. We remark that it is likely that the main result of the paper (Proposition 3.1) could be extended to more general cases. In particular, the third hypothesis is required in our proof but we believe that it is only a technical requirement that could be removed or at least weakened.

**2.2. Random Medium with Long-range Correlation.** In this section we present three processes $\nu$ that satisfy the conditions that we have imposed on the medium fluctuations.

**Fractional Ornstein Uhlenbeck medium.** The fractional Ornstein Uhlenbeck (OU) process $\tilde{\nu}(z)$ is defined by

$$\tilde{\nu}(z) := W^H(z) - \frac{1}{l_c}\int_{-\infty}^{z} e^{\frac{y-z}{l_c}} W^H(y)dy\,, \tag{2.6}$$

where $W^H$ is a fractional Brownian motion with Hurst index $H \in (1/2, 1)$. The fractional OU process is a zero-mean, stationary, Gaussian process and its autocorrelation function is given by

$$\tilde{\phi}_0(z) = -\frac{1}{2}|z|^{2H} + \frac{1}{4l_c}\int_{-\infty}^{\infty} e^{-\frac{|y|}{l_c}}|z+y|^{2H}dy\,.$$

The large-$z$ behavior of the autocorrelation function is (2.4) with

$$\alpha = 2-2H \ \text{ and } \tilde{c}_\alpha = H(2H-1)l_c^2\,.$$

It is possible to simulate paths of this process using the methods described in [2].

However, the fractional OU process $\tilde{\nu}$ is not bounded, nor differentiable. Let us consider the regularized process $\nu$ defined by

$$\nu(z) = \big[K * (T(\tilde{\nu}))\big](z) = \int K(z-y)T(\tilde{\nu}(y))dy\,, \tag{2.7}$$

where $T$ is a smooth, bounded, and odd real-valued function, such as arctan, and $K$ is a smooth convolution kernel, such as a Gaussian kernel. Applying Lemma A.1 we obtain that the process $\nu$ satisfies Assumption 1.

**Fractional white noise medium.** As a second example we can consider the model

$$\tilde{\nu}(z) := W^H(z) - W^H(z+l_c)\,, \tag{2.8}$$

where $W_H$ is a fractional Brownian motion. The large-$z$ behavior of the autocorrelation function is (2.4) with

$$\alpha = 2 - 2H \text{ and } \tilde{c}_\alpha = H(2H-1)l_c^2 .$$

Next, we regularize this process using Lemma A.1 as in (2.7) .

**Binary medium.** Here we construct a process corresponding to a binary medium, the process $\tilde{\nu}$ is then stepwise constant and takes values $\pm\sigma$ over intervals with random lengths. We denote by $(l_j)_{j\geq 0}$ the lengths of these intervals and by $(n_j)_{j\geq 0}$ the values taken by the process over each elementary interval. The process $\tilde{\nu}(z)$ is defined by

$$\tilde{\nu}(z) := n_{N_z} \text{ where } N_z = \sup\{n \geq 0,\, L_n \leq z\} , \tag{2.9}$$

where $L_0 = 0$ and $L_{n+1} = L_n + l_n$. The random variables $n_j$ are independent and identically distributed with the distribution

$$\mathbb{P}(n_j = \pm\sigma) = \frac{1}{2} .$$

The random variables $l_j$ are independent and identically distributed and their distribution has the probability density function (pdf)

$$p_{l_1}(z) = (1+\alpha)\frac{l_c^{1+\alpha}}{z^{2+\alpha}}\mathbf{1}_{[l_c,\infty)}(z) . \tag{2.10}$$

Note that it is very easy to simulate the random variable $l_1$, since $l_c U^{-1/(1+\alpha)}$ has the pdf (2.10) if $U$ is uniformly distributed over $[0,1]$. The average length of the random interval is

$$\mathbb{E}[l_1] = \frac{1+\alpha}{\alpha} l_c ,$$

while the variance of $l_1$ is infinite. The process $\tilde{\nu}$ is bounded and has mean zero, but it is not stationary. However, using renewal theory [14, Chap. 11], one can show that the distribution of the process $(\tilde{\nu}(y+z))_{z\geq 0}$ converges to a stationary distribution when $y \to \infty$ and that the autocorrelation function of $\tilde{\nu}$ satisfies

$$\mathbb{E}[\tilde{\nu}(y)\tilde{\nu}(y+z)] \stackrel{y\to\infty}{\longrightarrow} \tilde{\phi}_0(z) \tag{2.11}$$

$$\tilde{\phi}_0(z) \;=\; \sigma^2 \left[ \frac{1}{1+\alpha}\frac{l_c^\alpha}{z^\alpha}\mathbf{1}_{[l_c,\infty)}(z) + \Big(1 - \frac{\alpha}{\alpha+1}\frac{z}{l_c}\Big)\mathbf{1}_{[0,l_c)}(z)\right] , \tag{2.12}$$

which is of the form (2.4) with $\tilde{c}_\alpha = \sigma^2 l_c^\alpha/(1+\alpha)$.

It is also possible to make the process stationary by simply modifying the statistical distribution of the length of the first interval: If the random lengths $(l_j)_{j\geq 1}$ are independent and identically distributed according to the distribution with the pdf (2.10), and if $l_0$ is independent of the $(l_j)_{j\geq 1}$ and has the distribution with the pdf:

$$p_{l_0}(z) = \frac{\mathbb{P}(l_1 > z)}{\mathbb{E}[l_1]} = \frac{\alpha}{\alpha+1}\frac{1}{l_c}\mathbf{1}_{[0,l_c)}(z) + \frac{\alpha}{\alpha+1}\frac{l_c^\alpha}{z^{1+\alpha}}\mathbf{1}_{[l_c,\infty)}(z) ,$$

then the process $\tilde{\nu}$ is bounded, zero-mean and stationary, and its autocorrelation function $\mathbb{E}[\tilde{\nu}(y)\tilde{\nu}(y+z)]$ is (2.12) for any $y$.

Note that the process $\tilde{\nu}$ is bounded but not differentiable. If we consider a regularized version $\nu(z) = K * \tilde{\nu}(z)$, where $K$ is for instance a Gaussien kernel, then $\nu$ is differentiable with $\|\nu^{(j)}\|_\infty \leq \|K^{(j)}\|_1\sigma$ and $\nu$ satisfies Assumption 1.

**2.3. The Propagating Modes.** We consider the right- and left-going waves defined in terms of the local impedance and moving with the local sound speed:

$$\begin{bmatrix} A^\varepsilon(t,z) \\ B^\varepsilon(t,z) \end{bmatrix} := \begin{bmatrix} \zeta^{\varepsilon\,-1/2}(z)p(t,z) + \zeta^{\varepsilon\,1/2}(z)u(t,z) \\ -\zeta^{\varepsilon\,-1/2}(z)p(t,z) + \zeta^{\varepsilon\,1/2}(z)u(t,z) \end{bmatrix} . \tag{2.13}$$

The local impedance is

$$\zeta^\varepsilon(z) := \sqrt{K(z)\rho(z)} = \frac{\bar{\zeta}}{\sqrt{1+\varepsilon\nu(z/\varepsilon^2)}} .$$

The mode amplitudes satisfy

$$\frac{\partial}{\partial z}\begin{bmatrix} A^\varepsilon \\ B^\varepsilon \end{bmatrix} = -\frac{1}{c^\varepsilon(z)}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\frac{\partial}{\partial t}\begin{bmatrix} A^\varepsilon \\ B^\varepsilon \end{bmatrix} + \frac{\zeta^{\varepsilon\prime}(z)}{2\zeta^\varepsilon(z)}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} A^\varepsilon \\ B^\varepsilon \end{bmatrix} . \tag{2.14}$$

Here $\zeta^{\varepsilon\prime}$ is the $z$-derivative of $\zeta^\varepsilon$ and

$$\frac{\zeta^{\varepsilon\prime}(z)}{\zeta^\varepsilon(z)} = -\frac{1}{2\varepsilon}\frac{\nu'(z/\varepsilon^2)}{1+\varepsilon\nu(z/\varepsilon^2)} .$$

The local sound speed is

$$c^\varepsilon(z) := \sqrt{K(z)/\rho(z)} = \frac{\bar{c}}{\sqrt{1+\varepsilon\nu(z/\varepsilon^2)}} . \tag{2.15}$$

This system is completed with an initial condition corresponding to a right-going wave that is incoming from the homogeneous half-space $z < 0$ and is impinging on the random medium in $[0, L]$,

$$A^\varepsilon(t,z) = f\Big(\frac{t-z}{\varepsilon^2}\Big), \qquad B^\varepsilon(t,z) = 0 , \qquad t < 0 . \tag{2.16}$$

The source pulse function $f$ is compactly supported in the interval $(-T_0, T_0)$. Equation (2.14) clearly exhibits the two important aspects of the propagation mechanisms. The first term on the right describes transport along the random characteristics with the local sound speed $c^\varepsilon(z)$. The second term on the right describes coupling between the right- and left-going modes, which is proportional to the derivative $\zeta^{\varepsilon\prime}$ of the impedance.

Before considering the random medium with long-range correlation, we briefly recall the standard O'Doherty-Anstey (ODA) theory that describes the propagating pulse when the medium has rapidly decaying correlations. The effective equation for the wave front has in this case been obtained by several authors [5, 6, 10, 15, 23, 27]. The pulse propagation is characterized by a random time shift and a deterministic spreading, that are of the same order. The random time shift is described in terms of a standard Brownian motion, while the deterministic spreading is described by a pseudo-differential operator. If, additionally, the correlation length of the medium is smaller than the typical wavelength, then the pseudo-differential operator can be reduced to a second-order diffusion.

**3. Asymptotic Analysis of the Wave Front.**

**3.1. Statement of the Main Result.** We now state the main result that characterizes the wave front transmitted through a random medium with long-range correlation.

PROPOSITION 3.1. *Let us introduce the random travel time*

$$\tau_0^\varepsilon(z) := \frac{z}{\bar{c}} + \frac{\varepsilon}{2\bar{c}} \int_0^z \nu\left(\frac{y}{\varepsilon^2}\right) dy \,. \tag{3.1}$$

*1. Under Assumption 1, the wave front observed in the random frame moving with the random travel time*

$$A^\varepsilon\left(\tau_0^\varepsilon(z) + \varepsilon^2\tau, z\right) , \qquad z > 0 \;, \tag{3.2}$$

*converges in distribution as* $\varepsilon \to 0$ *to the deterministic profile*

$$a(\tau, z) := \frac{1}{2\pi} \int \exp\Big( - i\omega\tau - \frac{\gamma(\omega)\omega^2}{8\bar{c}^2} z - i\frac{\gamma^{(s)}(\omega)\omega^2}{8\bar{c}^2} z\Big) \hat{f}(\omega) d\omega \,, \tag{3.3}$$

*where* $\hat{f}(\omega)$ *is the Fourier transform of the initial pulse and*

$$\gamma(\omega) := 2\int_0^\infty \phi_0(z) \cos\Big(\frac{2\omega z}{\bar{c}}\Big) dz \,, \tag{3.4}$$

$$\gamma^{(s)}(\omega) := 2\int_0^\infty \phi_0(z) \sin\Big(\frac{2\omega z}{\bar{c}}\Big) dz \,. \tag{3.5}$$

*2. Under Assumption 1, the expectation of the random travel time* $\tau_0^\varepsilon(z)$ *is* $z/\bar{c}$ *and its variance is*

$$\mathrm{Var}\big(\tau_0^\varepsilon(z)\big) = \frac{\varepsilon^{2(1+\alpha)}}{\bar{c}^2} \frac{c_\alpha}{2(1-\alpha)(2-\alpha)} z^{2-\alpha} + o(\varepsilon^{2(1+\alpha)}) \,,$$

*as* $\varepsilon \to 0$.

*3. With some additional technical hypotheses that ensure that the integral of* $\nu$ *satisfies a non-central limit theorem, the random travel time* $\tau_0^\varepsilon(z)$ *has the distribution of*

$$\frac{z}{\bar{c}} + \frac{\varepsilon^{1+\alpha}}{\bar{c}} \sqrt{\frac{c_\alpha}{2(1-\alpha)(2-\alpha)}} W^H(z) + o(\varepsilon^{1+\alpha}) \,,$$

*as* $\varepsilon \to 0$*, where* $W^H(z)$ *is a fractional Brownian motion with Hurst index* $H = 1 - \alpha/2$.

The third point of this proposition was established in [22] for a certain class of subordinated Gaussian processes. We extend this result in Appendix B so that the first two models introduced in Subsection 2.2 satisfy the non-central limit theorem. The third model should also satisfy the non-central limit theorem but the proof requires some more work.

We see from the first point of this proposition that the frequency-dependent decay rate

$$\frac{\gamma(\omega)\omega^2}{8\bar{c}^2} \,, \tag{3.6}$$

of the wave front in (3.3) is always nonnegative because $\gamma(\omega)$ is the power spectral density of the stationary fluctuations $\nu(z)$ of the random medium.

The term $\exp[-i\gamma^{(s)}(\omega)\omega^2 z/(8\bar{c}^2)]$ in (3.3) is a frequency-dependent phase modulation and $\gamma^{(s)}(\omega)$ is conjugate to $\gamma(\omega)$. This shows that the transmitted wave front when centered with respect to the random travel time correction propagates in a dispersive effective medium with frequency-dependent wave number, given by

$$k(\omega) = \frac{\omega}{\bar{c}} - \varepsilon^2 \frac{\gamma^{(s)}(\omega)\omega^2}{8\bar{c}^2},$$

up to higher-order terms.

Proposition 3.1 shows that the transmitted wave front in the random medium is modified in two ways compared to propagation in a homogeneous one. First, its arrival time at the end of the slab $z = L$ has a small random component of order $\varepsilon^{1+\alpha}$. Its statistical distribution in terms of a fractional Brownian motion was already obtained in [22]. Remember, however, that the pulse width is of order $\varepsilon^2$, which means that the random time delay is large compared to the pulse width, moreover, it becomes relatively larger as $\alpha$ decreases. Second, if we observe the wave front near its random arrival time, then we see a pulse profile that, to leading order, is deterministic and is the original pulse shape convolved with a deterministic kernel that depends on the second-order statistics of the medium through the autocorrelation function of $\nu$:

$$a(\tau, z) = [\mathcal{H}(\cdot, z) * f](\tau).$$

The convolution kernel is given by

$$\mathcal{H}(\tau, z) = \frac{1}{2\pi} \int \exp\Big( -i\omega\tau - \frac{\gamma(\omega)\omega^2}{8\bar{c}^2} z - i\frac{\gamma^{(s)}(\omega)\omega^2}{8\bar{c}^2} z \Big) d\omega.$$

From the integral equation formulation of the wave front problem that we derive next in Section 4, we can see that only second-order scattering events contribute in the asymptotic analysis. This explains why only second-order statistics of the fluctuations are involved.

**3.2. The Random Time Shift.** In this subsection we analyze the asymptotic behavior of the random travel time $\tau_0^\varepsilon(z)$ defined by (3.1). We note that $\tau_0^\varepsilon(z)$ is not the travel time along the random characteristics of (2.14), which by (2.15) is given by

$$\tau^\varepsilon(z) := \int_0^z \frac{1}{c^\varepsilon(y)} dy = \frac{z}{\bar{c}} + \frac{\varepsilon}{2\bar{c}} \int_0^z \nu\left(\frac{y}{\varepsilon^2}\right) dy - \frac{\varepsilon^2}{8\bar{c}} \int_0^z \nu^2\left(\frac{y}{\varepsilon^2}\right) dy + O(\varepsilon^3), \quad (3.7)$$

corresponding to the first arrival time to depth $z$ for a point source at the surface. The $\varepsilon^2$ term in (3.7) is not present in (3.1). It is one of the results of Proposition 3.1 that the travel time of the wave front has the form (3.1). It implies that the stable wave front arrives with the delay

$$\Delta\tau^\varepsilon(z) := \tau_0^\varepsilon(z) - \tau^\varepsilon(z)$$

after the arrival of the leading edge which arrives at the random time $\tau^\varepsilon(z)$. The mean delay is

$$\mathbb{E}\big[\Delta\tau^\varepsilon(z)\big] = \frac{\varepsilon^2}{8\bar{c}} \mathbb{E}\Big[\int_0^z \nu^2\left(\frac{y}{\varepsilon^2}\right) dy\Big] + O(\varepsilon^3) = \varepsilon^2 \frac{\mathbb{E}[\nu(0)^2] z}{8\bar{c}} + O(\varepsilon^3),$$

The variance of the delay satisfies

$$\begin{aligned}\mathrm{Var}\Big(\frac{\Delta\tau^\varepsilon(z)}{\varepsilon^2}\Big) &= \frac{1}{64\bar{c}^2}\int_0^z\int_0^z \mathbb{E}\Big[\nu^2\Big(\frac{y_1}{\varepsilon^2}\Big)\nu^2\Big(\frac{y_2}{\varepsilon^2}\Big)\Big] \\ &\qquad -\mathbb{E}\Big[\nu^2\Big(\frac{y_1}{\varepsilon^2}\Big)\Big]\mathbb{E}\Big[\nu^2\Big(\frac{y_2}{\varepsilon^2}\Big)\Big]dy_1dy_2 + O(\varepsilon) \\ &\stackrel{\varepsilon\to 0}{\longrightarrow} 0\,,\end{aligned}$$

by (2.5) and the dominated convergence theorem. This shows that the delay is to leading order *deterministic* and given by $\varepsilon^2\mathbb{E}[\nu(0)^2]z/(8\bar{c})$, hence it is on the scale of the source pulse support. The delay is caused by a gradual delay of the pulse due to the scattering events that transform the pulse shape as it moves into the medium.

The random travel time $\tau_0^\varepsilon(z)$ has mean $z/\bar{c}$ and variance

$$\mathrm{Var}(\tau_0^\varepsilon(z)) = \frac{\varepsilon^2}{4\bar{c}^2}\mathbb{E}\left[\Big(\int_0^z \nu\Big(\frac{y}{\varepsilon^2}\Big)dy\Big)^2\right] = \frac{\varepsilon^2}{4\bar{c}^2}\int_0^z\int_0^z \phi_0\Big(\frac{y-x}{\varepsilon^2}\Big)dydx\,.$$

Using (2.4) the variance has the following asymptotic behavior as $\varepsilon \to 0$:

$$\frac{1}{\varepsilon^{2(1+\alpha)}}\mathrm{Var}(\tau_0^\varepsilon(z)) \stackrel{\varepsilon\to 0}{\longrightarrow} \frac{c_\alpha}{4\bar{c}^2}\int_0^z\int_0^z |y-x|^{-\alpha}dydx = \frac{c_\alpha}{2(1-\alpha)(2-\alpha)\bar{c}^2}z^{2-\alpha}\,.$$

This proves the second point of Proposition 3.1. By using a non-central limit theorem applied to the antiderivative of the fluctuation process $\nu(z)$ we have that

$$\frac{1}{\varepsilon^{1+\alpha}}\left(\tau_0^\varepsilon(z) - \frac{z}{\bar{c}}\right) = \frac{1}{2\varepsilon^\alpha \bar{c}}\int_0^z \nu\left(\frac{y}{\varepsilon^2}\right)dy$$

converges in distribution as $\varepsilon \to 0$ to

$$\sqrt{\frac{c_\alpha}{2(1-\alpha)(2-\alpha)\bar{c}^2}}W^H(z)\,, \tag{3.8}$$

where $W^H(z)$ is a fractional Brownian motion with Hurst index $H = 1-\alpha/2$. This characterizes the fluctuations in the arrival time of the stable wave front around the deterministic arrival time $z/\bar{c}$ associated with the homogenized medium. There exist different versions of this non-central limit theorem with different hypotheses for $\nu$, but only subordinated Gaussian models and linear filters have been treated in detail [12, 30]. In particular, the non-central limit theorem has been established in [22] in the case in which $\nu$ is of the form $T(\tilde{\nu}(z))$ where $\tilde{\nu}$ is a Gaussian process with long-range correlation and $T$ is a bounded function. In Appendix B we extend this result to show that the examples of Subsection 2.2 satisfy the non-central limit theorem.

**3.3. The Deterministic Pulse Deformation.** In this section we analyze the main properties of the effective equation for the wave front: The important function affecting the dynamics is the Fourier transform (3.4-3.5) of the positive lag part of the autocorrelation function of the random fluctuations of the medium. We have stated that $A^\varepsilon(\tau_0^\varepsilon(z)+\varepsilon^2\tau, z)$ converges to $a$ given by (3.3). By taking an inverse Fourier transform, it is possible to identify the partial differential equation (PDE) satisfied by $a$:

$$\frac{\partial a}{\partial z} = \mathcal{L}a\,, \tag{3.9}$$

where $\mathcal{L}$ is a pseudo-differential operator that describes the deterministic pulse deformation:

$$\mathcal{L} = \mathcal{L}_r + \mathcal{L}_i \,, \tag{3.10}$$

$$\int_{-\infty}^{\infty} \mathcal{L}_r a(\tau) e^{i\omega\tau} d\tau = -\frac{\gamma(\omega)\omega^2}{8\bar{c}^2} \int_{-\infty}^{\infty} a(\tau) e^{i\omega\tau} d\tau \,, \tag{3.11}$$

$$\int_{-\infty}^{\infty} \mathcal{L}_i a(\tau) e^{i\omega\tau} d\tau = -\frac{i\gamma^{(s)}(\omega)\omega^2}{8\bar{c}^2} \int_{-\infty}^{\infty} a(\tau) e^{i\omega\tau} d\tau \,. \tag{3.12}$$

The PDE (3.9) is completed with the initial condition $a(\tau, z = 0) = f(\tau)$.

The first qualitative property satisfied by the pseudo-differential operator $\mathcal{L}$ is that it preserves the causality. Indeed, in the time domain, we can write

$$\mathcal{L}a(\tau) = \left[\frac{1}{8\bar{c}}\phi_0\left(\frac{\bar{c}\tau}{2}\right)\mathbf{1}_{[0,\infty)}(\tau)\right] * \left[\frac{\partial^2 a}{\partial \tau^2}(\tau)\right] = \frac{1}{8\bar{c}}\int_0^{\infty} \phi_0\left(\frac{\bar{c}s}{2}\right)\frac{\partial^2 a}{\partial \tau^2}(\tau - s) ds \,.$$

The indicator function $\mathbf{1}_{[0,\infty)}$ is essential to interpret correctly the convolution. If $a$ is vanishing for $\tau < 0$, then $\mathcal{L}a$ is also vanishing for $\tau < 0$.

The pseudo-spectral operator $\mathcal{L}$ can be divided into two parts as (3.10). The first component $\mathcal{L}_r$, as pointed out in Subsection 3.1 after (3.6), is a frequency-dependent attenuation which can be interpreted as an effective diffusion operator. However, $\mathcal{L}_r$ does not behave like a second-order diffusion $\partial_\tau^2$ as we discuss below. The second component $\mathcal{L}_i$ is an effective dispersion operator, since it preserves energy. There is an interesting regime that leads to explicit formula. This is the regime in which the typical wavenumber $\omega/\bar{c}$ of the input pulse is such that $\omega l_c/\bar{c} \ll 1$. Using (2.4) we then find

$$\frac{\gamma(\omega)\omega^2}{\bar{c}^2} = c_\alpha \frac{\sqrt{\pi}\Gamma(\frac{1}{2} - \frac{\alpha}{2})}{\Gamma(\frac{\alpha}{2})} \frac{|\omega|^{1+\alpha}}{\bar{c}^{1+\alpha}} \,, \qquad \frac{\gamma^{(s)}(\omega)\omega^2}{\bar{c}^2} = c_\alpha \frac{\sqrt{\pi}\Gamma(1 - \frac{\alpha}{2})}{\Gamma(\frac{1}{2} + \frac{\alpha}{2})} \frac{|\omega|^{1+\alpha}}{\bar{c}^{1+\alpha}} \mathrm{sgn}(\omega) \,. \tag{3.13}$$

This shows that the wave propagation in random media with a long-range correlation exhibits frequency-dependent attenuation that is characterized by a power law with an exponent ranging from 1 to 2 that is related to the power decay rate of the autocorrelation function of the medium fluctuations. The frequency-dependent attenuation is associated with a frequency-dependent phase. This ensures that causality is respected.

Although we do not use the limit theorems presented in [21], the results obtained in this paper are strongly connected to them. In [21] the solutions of ordinary differential equations driven by random and rapidly varying coefficients with long-range correlation are studied. It is shown that, when the solutions are observed at a particular scale that depends on the power decay rate of the autocorrelation function, then they usually converge to the solutions of stochastic differential equations driven by fractional Brownian motions. This result is in agreement with the behavior of the travel time that we have exhibited here. It is also shown in [21] that the presence of periodic components in the random ordinary differential equations can have a dramatic effect. In particular, it modifies the scale at which the solutions should be observed to obtain a convergence, and it also affects the type of limit equations, which are now driven by standard Brownian motions. This result is in agreement with the type of pulse deformation that we have obtained.

**3.4. The Mean-field Approach.** The derivation of the effective equation for the wave front is based on an integral representation of the wave front and the application of a stochastic limit theorem. Our approach gives more precise results than a mean-field theory, as we discuss now. Let us observe the coherent wave front (or mean field) in the frame moving with the sound speed $\bar{c}$ of the effective medium:

$$a^{\varepsilon}_{\rm coh}(\tau, z) := \mathbb{E}\left[A^{\varepsilon}\left(\frac{z}{\bar{c}} + \varepsilon^2 \tau, z\right)\right] . \tag{3.14}$$

The analysis of the coherent field exhibits an additional frequency-dependent decay that originates from the averaging with respect to the random time delay. This term is strong as it becomes of order one for a small propagation distance, of order $\varepsilon^{\frac{2-2\alpha}{2-\alpha}}$. More precisely, we have the following result:

LEMMA 3.2.

$$a^{\varepsilon}_{\rm coh}\left(\tau, \varepsilon^{\frac{2-2\alpha}{2-\alpha}} z\right) \stackrel{\varepsilon \to 0}{\longrightarrow} a_{\rm coh}(\tau, z) ,$$

*where the asymptotic mean field is*

$$a_{\rm coh}(\tau, z) = \frac{1}{2\pi} \int \exp\Big( - i\omega\tau - \frac{c_\alpha \omega^2}{4(1-\alpha)(2-\alpha)\bar{c}^2} z^{2-\alpha}\Big) \hat{f}(\omega) d\omega . \tag{3.15}$$

*Proof.* Using Proposition 3.1 we obtain the expression of the asymptotic mean field

$$a_{\rm coh}(\tau, z) = \mathbb{E}\Big[f\Big(\tau + \frac{1}{\bar{c}}\sqrt{\frac{c_\alpha}{2(1-\alpha)(2-\alpha)}} W^H(z)\Big)\Big] .$$

Since $W^H(z)$ has a zero-mean Gaussian distribution with variance $z^{2(1-\alpha/2)}$, the expectation reads

$$a_{\rm coh}(\tau, z) = \frac{1}{\sqrt{2\pi z^{2(1-\alpha/2)}}} \int f\Big(\tau + \frac{1}{\bar{c}}\sqrt{\frac{c_\alpha}{2(1-\alpha)(2-\alpha)}} w\Big) \exp\Big( - \frac{w^2}{2 z^{2(1-\alpha/2)}}\Big) dw .$$

This integral is the convolution of $f$ with a Gaussian kernel, which can be written as (3.15). □

The partial differential equation satisfied by the asymptotic mean field is:

$$\frac{\partial a_{\rm coh}}{\partial z} = \frac{c_\alpha}{4(1-\alpha)\bar{c}^2} z^{1-\alpha} \frac{\partial^2 a_{\rm coh}}{\partial \tau^2}$$

Thus, the coherent wave is described by an anomalous diffusion equation. If, for instance, the initial pulse has the Gaussian shape:

$$f(\tau) = q_0 \exp\Big( - \frac{\tau^2}{2T_0^2}\Big) ,$$

then the output coherent pulse is given by

$$a_{\rm coh}(\tau, z) = q(z) \exp\Big( - \frac{\tau^2}{2T(z)^2}\Big) ,$$

where the width of the coherent pulse increases as

$$T(z)^2 = T_0^2 + \frac{c_\alpha}{2(1-\alpha)(2-\alpha)\bar{c}^2} z^{2-\alpha} ,$$

and its amplitude $q(z)$ decays as

$$q(z) = q_0 \frac{T_0}{T(z)} .$$

This anomalous diffusion is strong, much stronger than the diffusion for the randomly centered wave front, but it is not physical as it is determined by the averaging with respect to the random time shift.

**4. Derivation of the Effective Equation for the Wave Front.** In this section we give the proof of the first point of Proposition 3.1, which goes along the same lines as the one given in [15] in the case where the autocorrelation function $\phi_0$ is integrable. We will first perform a series of transformations to rewrite the evolution equations of the modes by centering along the characteristic of the right-going mode. We will then obtain an upper-triangular system that can be integrated more easily. In a second step we will apply a limit theorem to this system to establish an effective equation for the wave front.

We introduce the characteristic random travel time (3.7) and consider the new reference frame

$$(z,t) \mapsto (\tau, s) \ , \quad \text{with } \tau = \tau^\varepsilon(z) \ \text{ and } s = \frac{t-\tau^\varepsilon(z)}{\varepsilon^2} \, , \tag{4.1}$$

which moves with the right-going mode $A^\varepsilon$ and is adjusted to be on the time scale of the incident pulse. In this new reference frame the equations for $(A^\varepsilon, B^\varepsilon)$ have the form

$$\frac{\partial}{\partial \tau} \begin{bmatrix} A^\varepsilon \\ B^\varepsilon \end{bmatrix} = \frac{1}{\varepsilon^2} \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} \frac{\partial}{\partial s} \begin{bmatrix} A^\varepsilon \\ B^\varepsilon \end{bmatrix} - \frac{1}{4\varepsilon} M^\varepsilon \Big( \frac{z^\varepsilon(\tau)}{\varepsilon^2} \Big) \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} A^\varepsilon \\ B^\varepsilon \end{bmatrix} , \tag{4.2}$$

where

$$M^\varepsilon(z) := \bar{c} \frac{\nu'(z)}{(1+\varepsilon \nu(z))^{3/2}} ,$$

and $z^\varepsilon(\tau)$ is the inverse function of the travel time $\tau^\varepsilon(z)$. This is a lower-triangular system that we can integrate. More precisely, the equation for $A^\varepsilon$ can be integrated for $\tau > 0$:

$$A^\varepsilon(s,\tau) = -\frac{1}{4\varepsilon} \int_0^\tau M^\varepsilon \Big( \frac{z^\varepsilon(y)}{\varepsilon^2} \Big) B^\varepsilon(s,y) dy + f(s) \, . \tag{4.3}$$

For $\tau \leq 0$, we simply have $A^\varepsilon(s,\tau) = f(s)$. The integrated form of the equation for $B^\varepsilon$ is

$$B^\varepsilon(s,\tau) = -\frac{\varepsilon^2}{2} \int_{-\infty}^s S_B^\varepsilon \Big( u, \tau + \frac{\varepsilon^2}{2}(s-u) \Big) du \, , \tag{4.4}$$

where

$$S_B^\varepsilon(s,\tau) := -\frac{1}{4\varepsilon} M^\varepsilon \Big( \frac{z^\varepsilon(\tau)}{\varepsilon^2} \Big) A^\varepsilon(s,\tau) \, . \tag{4.5}$$

The integral in (4.4) is over the infinite range $(-\infty, s)$. However, the initial conditions restrict $A^\varepsilon$ and $B^\varepsilon$ to be zero for $s < -T_0$ and $\tau = 0$. From equations (4.2) we then see that $A^\varepsilon$ and $B^\varepsilon$ are zero for $s < -T_0$ for any $\tau \geq 0$. Thus the integral with respect to $u$ in (4.4) is effectively limited to the range $(-T_0, s)$. If we now substitute the integral representation (4.4) for $B^\varepsilon$ into the one (4.3) for $A^\varepsilon$ we obtain

$$A^\varepsilon(s,\tau) = f(s) - \frac{1}{32}\int_0^\tau M^\varepsilon\Big(\frac{z^\varepsilon(y)}{\varepsilon^2}\Big) \times \int_{-T_0}^s M^\varepsilon\Big(\frac{z^\varepsilon(y+\varepsilon^2(s-u)/2)}{\varepsilon^2}\Big)A^\varepsilon\Big(u, y+\varepsilon^2\frac{s-u}{2}\Big)du\,dy\,. \tag{4.6}$$

This is the closed integral equation for the advancing front of the transmitted wave. We will apply the averaging theorem to a somewhat simplified version of this equation.

We first transform the integral equation (4.6) into a form that is asymptotically equivalent to it as $\varepsilon \to 0$ and that allows direct application of the averaging theorem.

From (4.6) we get the inequality

$$\sup_{\tau\in[0,\tau^\varepsilon(L)]}|A^\varepsilon(s,\tau)| \leq |f(s)| + \frac{M^2\tau^\varepsilon(L)}{32}\int_{-T_0}^s \sup_{\tau\in[0,\tau^\varepsilon(L)]}|A^\varepsilon\big(u,\tau\big)|du\,,$$

where $M = \bar{c}\|\nu'\|_\infty/(1-\varepsilon_0\|\nu\|_\infty)^{3/2}$ is an upper bound for $M^\varepsilon$ valid for any $\varepsilon < \varepsilon_0$. We also have $\tau^\varepsilon(L) \leq L/[\bar{c}(1-\varepsilon_0\|\nu\|_\infty)]$. Using Gronwall's lemma we then obtain for any $\varepsilon < \varepsilon_0$ and $T > 0$ the estimate

$$\sup_{\tau\in[0,\tau^\varepsilon(L)],s\leq T}|A^\varepsilon(s,\tau)| \leq e^{M_2 L(T+T_0)}\|f\|_\infty\,.$$

Here $M_2 = M^2/[32\bar{c}(1-\varepsilon_0\|\nu\|_\infty)]$. Substituting this estimate into (4.4) and (4.3), we get the further estimates

$$\sup_{\tau\in[0,\tau^\varepsilon(L)],s\leq T}|B^\varepsilon(s,\tau)| \leq \varepsilon K_{T,L}\,, \qquad \sup_{\tau\in[0,\tau^\varepsilon(L)],s\leq T}\Big|\frac{\partial A^\varepsilon}{\partial\tau}(s,\tau)\Big| \leq K_{T,L}\,,$$

where $K_{T,L}$ is a constant that depends only on $T$ and $L$. From the estimate for $\partial_\tau A^\varepsilon$, we see that we can replace the last term of the integral in (4.6) by $A^\varepsilon(u,y)$, with an error of order $\varepsilon^2$. After the change of variable $x = z^\varepsilon(y)$ we obtain the integral equation

$$A^\varepsilon(s,\tau) = f(s) - \frac{1}{32}\int_0^{z^\varepsilon(\tau)} M^\varepsilon\Big(\frac{x}{\varepsilon^2}\Big)\frac{1}{c^\varepsilon(x)} \times \int_{-T_0}^s M^\varepsilon\Big(\frac{z^\varepsilon(\tau^\varepsilon(x)+\varepsilon^2(s-u)/2)}{\varepsilon^2}\Big)A^\varepsilon\Big(u,\tau^\varepsilon(x)\Big)du\,dx\,. \tag{4.7}$$

Since the second derivative of $\nu$ is bounded, we have

$$\begin{aligned} M^\varepsilon\Big(\frac{z^\varepsilon(\tau^\varepsilon(x)+\varepsilon^2(s-u)/2)}{\varepsilon^2}\Big) &= M^\varepsilon\Big(\frac{x}{\varepsilon^2}+\bar{c}\frac{s-u}{2}\Big) + O(\varepsilon^2) \\ &= \bar{c}\nu'\Big(\frac{x}{\varepsilon^2}+\bar{c}\frac{s-u}{2}\Big) + O(\varepsilon)\,. \end{aligned}$$

We also have that

$$c^\varepsilon(x) = \bar{c} + O(\varepsilon)\,, \qquad z^\varepsilon(\tau) = \bar{c}\tau + O(\varepsilon)\,, \qquad \tau^\varepsilon(x) = x/\bar{c} + O(\varepsilon)\,,$$

uniformly in $x \in [0, L]$ and $\tau \in [0, L/[\bar{c}(1-\varepsilon_0\|\nu\|_\infty)]]$. Using once again the uniform bound on $\partial_\tau A^\varepsilon$, we see that

$$A^\varepsilon(u, \tau^\varepsilon(x)) = A^\varepsilon(u, x/\bar{c}) + O(\varepsilon)\,,$$

which allows us to simplify the integral equation (4.7) for $A^\varepsilon$,

$$A^\varepsilon(s,\tau) = f(s) - \frac{\bar{c}}{32}\int_0^{\bar{c}\tau} \nu'\Big(\frac{x}{\varepsilon^2}\Big)\int_{-T_0}^{s} \nu'\Big(\frac{x}{\varepsilon^2} + \bar{c}\frac{s-u}{2}\Big) A^\varepsilon\Big(u, \frac{x}{\bar{c}}\Big) du\, dx\,,$$

where we have neglected terms of order $\varepsilon$. We make the change of variable $x = \bar{c}y$, then this integral equation can be written as

$$A^\varepsilon(s,\tau) = f(s) - \frac{\bar{c}^2}{32}\int_0^{\tau} \nu'\Big(\bar{c}\frac{y}{\varepsilon^2}\Big)\int_{-T_0}^{s} \nu'\Big(\bar{c}\frac{y}{\varepsilon^2} + \bar{c}\frac{s-u}{2}\Big) A^\varepsilon\big(u, y\big) du\, dy\,.$$

In a functional form this equation becomes

$$A^\varepsilon(\cdot,\tau) = f(\cdot) + \int_0^\tau F\Big(\frac{y}{\varepsilon^2}\Big) A^\varepsilon(\cdot, y) dy\,, \tag{4.8}$$

where $F(y)$ is the random linear operator acting on functions $A(\cdot)$ with support in $(-T_0, \infty)$, defined by

$$[F(y)A](s) := -\frac{\bar{c}^2}{32}\nu'(\bar{c}y)\int_{-T_0}^{s} \nu'\Big(\bar{c}y + \bar{c}\frac{s-u}{2}\Big) A(u) du\,. \tag{4.9}$$

Using Assumption 1, the following averaging theorem holds.

PROPOSITION 4.1. *The solution $A^\varepsilon(\cdot,\tau)$ of the integral equation (4.8) converges as $\varepsilon \to 0$ in probability, as a process in the space of continuous functions, to the solution of the averaged integral equation*

$$\tilde{A}(\cdot,\tau) = f(\cdot) + \int_0^\tau \tilde{F}\tilde{A}(\cdot, y) dy\,, \tag{4.10}$$

*where $\tilde{F} = \mathbb{E}[F(y)]$, that is,*

$$[\tilde{F}A](s) = -\frac{\bar{c}^2}{32}\int_{-T_0}^{s} \mathbb{E}\left[\nu'(\bar{c}y)\nu'\Big(\bar{c}y + \bar{c}\frac{s-u}{2}\Big)\right] A(u) du\,. \tag{4.11}$$

The proof of this averaging theorem is given in Appendix C. If we denote by $\phi_1$ the autocorrelation function of the stationary random process $\nu'$,

$$\phi_1(x) := \mathbb{E}[\nu'(z)\nu'(z+x)]\,,$$

then the operator $\tilde{F}$ acting on functions $A(\cdot)$ with support in $(-T_0, \infty)$ has the form

$$\tilde{F}A(s) = -\frac{\bar{c}^2}{32}\int_{-T_0}^{s} \phi_1\Big(\frac{\bar{c}}{2}(s-u)\Big) A(u) du = -\frac{\bar{c}^2}{32}\int_0^{T_0+s} \phi_1\Big(\frac{\bar{c}}{2}u\Big) A(s-u) du\,.$$

This operator can also be written as a convolution independently of the point $-T_0$ defining the left end of support of $A$:

$$\tilde{F}A(s) = -\frac{\bar{c}^2}{32}\int_0^{\infty} \phi_1\Big(\frac{\bar{c}}{2}u\Big) A(s-u) du\,.$$

In the Fourier domain the convolution operator $\tilde{F}$ is the multiplication operator

$$\int_{-\infty}^{\infty} \tilde{F}A(s)e^{i\omega s}ds = -\frac{\bar{c}}{16}b_1\Big(\frac{2\omega}{\bar{c}}\Big)\int_{-\infty}^{\infty} A(s)e^{i\omega s}ds\,, \tag{4.12}$$

where

$$b_1(k) := \int_0^{\infty} \phi_1(x)e^{ikx}dx\,. \tag{4.13}$$

We will now rewrite $b_1$ in terms of the autocorrelation function of the stationary random process $\nu$. Let us define

$$b_0(k) := \int_0^{\infty} \phi_0(x)e^{ikx}dx\,, \tag{4.14}$$

where $\phi_0$ is the autocorrelation function (2.3) of $\nu$. First, we note that $\partial_x^2\phi_0(x) = \mathbb{E}[\nu(z)\nu''(z+x)]$. We also note that $\phi_0$ is independent of $z$, by stationarity, so that $0 = \partial_z\partial_x\phi_0(x) = \mathbb{E}[\nu(z)\nu''(z+x)]+\mathbb{E}[\nu'(z)\nu'(z+x)]$. As a result we have the identity

$$\phi_1(x) = -\phi_0''(x)\,. \tag{4.15}$$

By integration by parts we get

$$b_1(k) = -\int_0^{\infty} \phi_0''(x)e^{ikx}dx = -\left[\phi_0'(x)e^{ikx}\right]_0^{\infty} + ik\int_0^{\infty} \phi_0'(x)e^{ikx}dx\,.$$

Since $\phi_0$ is even and differentiable we have $\phi_0'(0) = 0$, therefore, the first term on the right side vanishes. Integrating by parts once again we obtain

$$b_1(k) = ik\left[\phi_0(x)e^{ikx}\right]_0^{\infty} + k^2\int_0^{\infty} \phi_0(x)e^{ikx}dx = -ik\phi_0(0) + k^2 b_0(k)\,. \tag{4.16}$$

Using (4.16) in (4.12) the linear operator $\tilde{F}$ is therefore given by

$$\int_{-\infty}^{\infty} \tilde{F}A(s)e^{i\omega s}ds = \left[\frac{i\omega}{8}\phi_0(0) - \frac{\omega^2}{4\bar{c}}b_0\Big(\frac{2\omega}{\bar{c}}\Big)\right]\int_{-\infty}^{\infty} A(s)e^{i\omega s}ds\,. \tag{4.17}$$

We have shown that the wave front converges to a deterministic pulse profile when it is observed in the frame moving to the right with the random local sound speed $c^\varepsilon(z)$. If we observe the wave front in the frame moving along $\tau_0^\varepsilon(z)$, then we have to account for the difference between the random characteristic travel times $\tau^\varepsilon(z)$ given by (3.7) and $\tau_0^\varepsilon(z)$ given by (3.1). The rescaled travel time correction is

$$\frac{1}{\varepsilon^2}\left(\tau^\varepsilon(z) - \tau_0^\varepsilon(z)\right) = -\frac{1}{8\bar{c}}\int_0^z \nu\Big(\frac{x}{\varepsilon^2}\Big)^2 dx + O(\varepsilon)\,.$$

As shown in Subsection 3.2 we have the following convergence in mean square sense and in probability as $\varepsilon \to 0$:

$$\frac{1}{\varepsilon^2}\left(\tau^\varepsilon(z) - \tau_0^\varepsilon(z)\right) \stackrel{\varepsilon\to 0}{\longrightarrow} -\frac{1}{8\bar{c}}\phi_0(0)z\,. \tag{4.18}$$

The deterministic correction $-\phi_0(0)z/(8\bar{c})$ cancels with the first term on the right in (4.17), when written in the time domain and used in (4.10). That is why the travel time fluctuation of the wave front is simply the fractional Brownian motion part of (4.18) in the limit $\varepsilon \to 0$. This completes the proof of the first point of Proposition 3.1.

**5. Numerical Simulations.** In this section we present the results of full numerical simulations of the wave equations with a random medium. We use a spectral code where the bandwidth is discretized into 2048 frequencies. We consider a binary medium as described in Section 2.2, with the parameters $\sigma = 0.1$ and $l_c = 0.02$. The effective density $\bar{\rho}$ and bulk modulus $\bar{K}$ are both equal to 1, so that the effective speed of sound is $\bar{c} = 1$. The output pulse profiles are observed at the propagation distance $L = 1000$. The initial pulse $f(t) = (1 - \frac{5}{2}t^2)\exp(-\frac{5t^2}{4})$ is the second derivative of a Gaussian with amplitude 1.

Here the variance of the random time shift predicted by the asymptotic theory is:

$$\mathrm{Var}(\tau_0(L)) = \frac{c_\alpha L^{2-\alpha}}{2(1-\alpha)(2-\alpha)} = \frac{\sigma^2 l_c^\alpha L^{2-\alpha}}{2(1-\alpha^2)(2-\alpha)},$$

which is equal to 2.73 for $\alpha = 0.75$, 19.88 for $\alpha = 0.5$, and 218.36 for $\alpha = 0.25$. The fact that the variance of the random time shift increases as $\alpha$ is reduced can be clearly seen in Figures 5.1-5.3, left plots. Intuitively this follows since a smaller $\alpha$ corresponds to a longer range of interactions in the medium fluctuations. In the right plots we have shown the transmitted pulses when we center them with respect to the random travel time $\tau_0^\varepsilon$ together with the theoretical prediction $a(\tau, z)$. Note the excellent fit with the theory.

The autocorrelation function of the fluctuations of the process $\nu$ is given by (2.12). As $l_c$ is relatively small, the expressions of the frequency-dependent attenuation and phase can be approximated by

$$\gamma(\omega)\omega^2 = \frac{\sigma^2 l_c^\alpha}{1+\alpha} \frac{\sqrt{\pi}\Gamma(\frac{1}{2} - \frac{\alpha}{2})}{\Gamma(\frac{\alpha}{2})} |\omega|^{1+\alpha},$$

$$\gamma^{(s)}(\omega)\omega^2 = \frac{\sigma^2 l_c^\alpha}{1+\alpha} \frac{\sqrt{\pi}\Gamma(1 - \frac{\alpha}{2})}{\Gamma(\frac{1}{2} + \frac{\alpha}{2})} |\omega|^{1+\alpha}\mathrm{sgn}(\omega).$$

The frequency-dependent attenuation, respectively phase, corresponds in the time domain to attenuation, respectively dispersion, which can be seen in the numerical results. In particular, the dispersion is responsible for the asymmetry of the transmitted pulse profile.

We finally mention that the results obtained with the binary medium without smoothing are almost undistinguishable from the results obtained with a smoothed version of the binary medium obtaiend with splines. This is an indication that the results obtained in the paper under the set of hypotheses listed in Assumption 1 could be extended to cases in which the third hypothesis (smoothness) is not included.

**6. Stable Propagation in Random Media with Short-Range Correlation.** In the previous sections we considered the case when the random medium has long-range correlation in the sense that the autocorrelation function decays as $z^{-\alpha}$ with $\alpha \in (0, 1)$, so that it is not integrable. Here we briefly consider the case when the medium has short-range correlation in the sense that the autocorrelation function is integrable but the integral $\int \phi_0(z)dz = 0$. As a first model for the process $\nu(z)$, we can use the derivative $\nu(z) = \mu'(z)$ of a smooth stationary process $\mu(z)$ with rapidly-decaying correlation function $\phi_1(z)$. We then have $\phi_0(z) = -\phi_1''(z)$ and $\int \phi_0(z)dz = 0$. We can also consider models for which the autocorrelation function has a power law decay of the form $z^{-\alpha}$ with $\alpha \in (1, 2)$. Particular examples of such processes are the fractional OU process (2.6) and the fractional white noise model (2.8) described in Section 2.2, with a fractional Brownian motion with Hurst index $H \in (0, 1/2)$, which

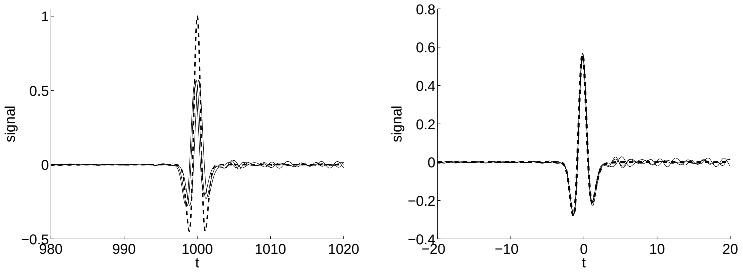


FIG. 5.1. *Left picture: wave front profiles for three different realizations of the random medium (solid lines) compared to the wave front profile obtained in homogeneous medium $\nu = 0$ (thick dashed line). Right picture: wave front profiles for three different realizations of the random medium (solid lines), centered in time, and compared to the theoretical asymptotic pulse profile (3.3) (thick dashed). Here* $\alpha = 0.75$.

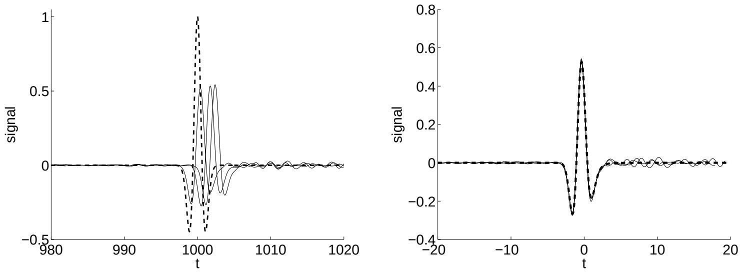


FIG. 5.2. *The same as in Figure 5.1, but* $\alpha = 0.5$ *here.*

gives $\alpha = 2 - 2H \in (1, 2)$ and $\int \phi_0(z)dz = 0$. Such processes are often referred to as anti-persistent since consecutive increments are negatively correlated, while they are positively correlated for $H > 1/2$ corresponding to a persistent process. Note also that the fractional Brownian motion has a modification whose sample paths are Hölder-continuous of any order in $(0, H)$ so that a small $H$ corresponds to a relatively rough process. We remark that modeling in terms of processes with $H < 1/2$ may be relevant in the context of the turbulent atmosphere while modeling in terms of processes corresponding to $H > 1/2$ may the more relevant model in the context of the multiscale crust of the earth.

The analysis of pulse propagation in random media with short-range correlation follows the one used for media with rapidly decaying correlation. One obtains that the pulse deformation is deterministic and described by the pseudo-differential operator (3.10). The original result is that the random travel time is negligible compared to the initial pulse width, since its variance at the scale of the pulse width is proportional to the integral of the autocorrelation function [15], which is here zero. Therefore, pulse propagation in random media with short-range correlation is fully stable and one obtains a frequency-dependent decay rate (3.6) of the wave front which can be expanded for small $\omega$ as $|\omega|^\gamma$ with $\gamma > 2$.

**7. Conclusion.** In this paper we have studied the propagation of an acoustic pulse in a one-dimensional random medium with long-range correlation. The fluctuations of the medium parameters are modeled by a random process with correlation decaying as $z^{-\alpha}$, $\alpha \in (0, 1)$. In the asymptotic regime where the amplitudes of the

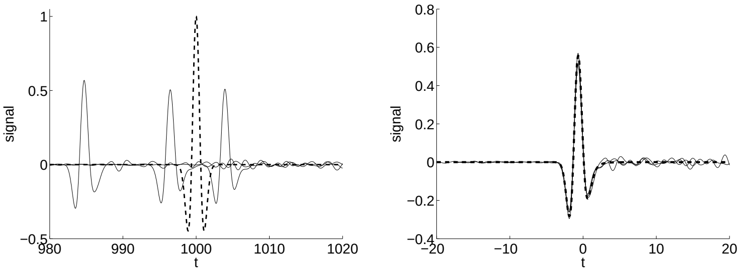


FIG. 5.3. *The same as in Figure 5.1, but $\alpha = 0.25$ here.*

fluctuations of the medium parameters are small and the propagation distance is large, we have obtained that the front wave is modified in two ways. It experiences a random time shift described in terms of a fractional Brownian motion and a deterministic spreading described by a pseudo-differential operator. In fact, the random travel time correction is large relative to the width of the propagating pulse. The pseudo-differential operator is characterized by a frequency-dependent attenuation that obeys a power law with the exponent $1 + \alpha$. This frequency-dependent attenuation is associated with a frequency-dependent phase, which ensures that causality is respected.

It would be interesting now to address the strong-fluctuations regime, in which the amplitude of the random fluctuations of the medium parameters is of order one compared to the average values. In this regime the approach used in this paper cannot be applied, and a Fourier approach could be of interest. The extension of the results to three-dimensional media is of course also of interest. The case of locally layered media could be dealt with the strategy adopted in [27] for random media with rapidly decaying correlation.

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**Appendix A. Regularization of Gaussian Processes.** Here we present a lemma that is used to prove that the examples proposed in Subsection 2.2 satisfy Assumption 1.

LEMMA A.1. *Let $\tilde{\nu}(z)$ be a zero-mean, stationary, Gaussian process such that $\tilde{\phi}_0(z) := \mathbb{E}[\tilde{\nu}(0)\tilde{\nu}(z)] \sim \tilde{c}_\alpha z^{-\alpha}$ as $z \to \infty$. Let $T$ be a smooth, bounded, odd, real-valued function. Let $K$ be a smooth convolution kernel. Then the process $\nu(z) := [K * (T(\tilde{\nu}))](z)$ satisfies Assumption 1. In particular, the large-$z$ behavior of the autocorrelation function $\phi_0$ of $\nu$ satisfies (2.4) with*

$$c_\alpha = \left( \int K^{*2}(s) ds \right) \left( \frac{1}{\sqrt{2\pi}} \int g T(g) e^{-\frac{g^2}{2}} dg \right)^2 \tilde{c}_\alpha \,. \tag{A.1}$$

*Proof.* The transformed process $\nu$ is bounded as well as its derivatives:

$$\|\nu^{(j)}\|_\infty \leq \|K^{(j)}\|_1 \|T\|_\infty \,, \quad j \geq 0 \,.$$

It is a zero-mean, stationary, random process. Its autocorrelation function is given by

$$\phi_0(z) = \int K^{*2}(s) \mathbb{E}\big[T(\tilde{\nu}(0)) T(\tilde{\nu}(z-s))\big] ds \,. \tag{A.2}$$

For any $z > 0$ we have

$$\mathbb{E}[T(\tilde{\nu}(0))T(\tilde{\nu}(z))] = \iint T(g_1)T(g_2)p_{\tilde{\phi}_0(z)}(g_1, g_2)dg_1 dg_2\,,$$

where $p_\phi$ is the pdf of a zero-mean Gaussian vector with correlation matrix $\begin{bmatrix} 1 & \phi \\ \phi & 1 \end{bmatrix}$:

$$p_\phi(g_1, g_2) = \frac{1}{2\pi\sqrt{1-\phi^2}} \exp\left(-\frac{g_1^2 + g_2^2 - 2\phi g_1 g_2}{2(1-\phi^2)}\right).$$

When $z \to \infty$, we have $\tilde{\phi}_0(z) \to 0$ and we can expand the value of the integral as

$$\begin{aligned}
\mathbb{E}\big[T(\tilde{\nu}(0))T(\tilde{\nu}(z))\big] &= \iint T(g_1)T(g_2)\frac{1}{2\pi}e^{-\frac{g_1^2+g_2^2}{2}}\left(1 + \tilde{\phi}_0(z)g_1g_2\right)dg_1dg_2 + o\big(\tilde{\phi}_0(z)\big) \\
&= \frac{1}{2\pi}\left(\int T(g)e^{-\frac{g^2}{2}}dg\right)^2 + \frac{\tilde{\phi}_0(z)}{2\pi}\left(\int gT(g)e^{-\frac{g^2}{2}}dg\right)^2 + o\big(\tilde{\phi}_0(z)\big) \\
&= \left(\frac{1}{\sqrt{2\pi}}\int gT(g)e^{-\frac{g^2}{2}}dg\right)^2 \tilde{c}_\alpha z^{-\alpha} + o(z^{-\alpha})\,.
\end{aligned}$$

Substituting into (A.2) and using the dominated convergence theorem gives

$$\phi_0(z) = \left(\int K^{*2}(s)ds\right)\left(\int gT(g)e^{-\frac{g^2}{2}}dg\right)^2 \tilde{c}_\alpha z^{-\alpha} + o(z^{-\alpha})\,.$$

It remains to show the fourth-order moment property (2.5) for

$$M(z) = \mathbb{E}\big[\nu(y_1)\nu(y_2)\nu(y_3+z)\nu(y_4+z)\big]\,. \tag{A.3}$$

We have

$$M(z) := \iint \prod_{j=1}^{4} K(y_j - z_j)\mathbb{E}\big[T(\tilde{\nu}(z_1))T(\tilde{\nu}(z_2))T(\tilde{\nu}(z+z_3))T(\tilde{\nu}(z+z_4))\big]dz_1\cdots dz_4\,. \tag{A.4}$$

Let us consider $z_j$, $j = 1, \ldots, 4$, such that $z_1 \neq z_2$ and $z_3 \neq z_4$. For $z$ large enough, the four points $z_1, z_2, z_3 + z, z_4 + z$ are distinct and

$$\begin{aligned}
\mathbb{E}\big[T(\tilde{\nu}(z_1))T(\tilde{\nu}(z_2))T(\tilde{\nu}(z+z_3))T(\tilde{\nu}(z+z_4))\big] &= \frac{1}{4\pi^2\sqrt{\det C(z_1, z_2, z_3+z, z_4+z)}} \\
\times \iint \prod_{j=1}^{4} T(\tilde{\nu}_j) &\exp\left(-\frac{\tilde{\nu}^t C^{-1}(z_1, z_2, z_3+z, z_4+z)\tilde{\nu}}{2}\right)d\tilde{\nu}_1\cdots d\tilde{\nu}_4\,,
\end{aligned}$$

where we have used vector notation in the exponent and the $t$ superscript means transpose. The entries of the $4\times 4$ matrix $C(z_1, z_2, z_3, z_4)$ are $C_{ij}(z_1, z_2, z_3, z_4) = \tilde{\phi}_0(z_i - z_j)$. Therefore we have

$$C(z_1, z_2, z_3+z, z_4+z) \overset{z\to\infty}{\longrightarrow} \begin{bmatrix} \tilde{\phi}_0(0) & \tilde{\phi}_0(z_1-z_2) & 0 & 0 \\ \tilde{\phi}_0(z_1-z_2) & \tilde{\phi}_0(0) & 0 & 0 \\ 0 & 0 & \tilde{\phi}_0(0) & \tilde{\phi}_0(z_3-z_4) \\ 0 & 0 & \tilde{\phi}_0(z_3-z_4) & \tilde{\phi}_0(0) \end{bmatrix},$$

which shows that

$$\mathbb{E}\big[T(\tilde{\nu}(z_1))T(\tilde{\nu}(z_2))T(\tilde{\nu}(z+z_3))T(\tilde{\nu}(z+z_4))\big] \overset{z\to\infty}{\longrightarrow} \mathbb{E}\big[T(\tilde{\nu}(z_1))T(\tilde{\nu}(z_2))\big]\mathbb{E}\big[T(\tilde{\nu}(z_3))T(\tilde{\nu}(z_4))\big] .$$

Substituting into (A.4) and using the dominated convergence theorem gives the fourth-order moment property (2.5) for the process $\nu(z)$. ☐

**Appendix B. The Non-central Limit Theorem.** We recall the version of the non-central limit theorem presented in [22]: If $\nu(z) = T(\tilde{\nu}(z))$, where $T$ is an odd, bounded $\mathcal{C}^\infty$-function and $\tilde{\nu}(z)$ is a zero-mean, stationary, Gaussian process whose autocorrelation function decays as $c_\alpha z^{-\alpha}$ as $z \to \infty$, then

$$\mathcal{N}^\varepsilon(z) := \frac{1}{\varepsilon^\alpha}\int_0^z \nu\big(\frac{y}{\varepsilon^2}\big)dy \tag{B.1}$$

converges in distribution as a continuous process as $\varepsilon \to 0$ to

$$\mathcal{N}(z) := \sqrt{\frac{c_\alpha}{2(1-\alpha)(2-\alpha)}}W^H(z) \tag{B.2}$$

where $W^H$ is a fractional Brownian motion with Hurst index $H = 1-\alpha/2$. The following lemma combined with this result shows that the first two models of Subsection 2.2 satisfy the non-central limit theorem.

Lemma B.1. *If a bounded process $\nu$ is such that $\mathcal{N}^\varepsilon(z)$ converges in distribution as a continuous process to $\mathcal{N}(z)$, and if $K$ is a smooth non-negative valued function such that $\int K(y)dy = 1$ and $\int |y|K(y)dy < \infty$, then the process $\nu_K(z) := K * \nu(z)$ is such that $\mathcal{N}_K^\varepsilon(z)$ (defined as (B.1) in terms of the process $\nu_K$) converges in distribution as a continuous process to $\mathcal{N}(z)$.*

*Proof.* We have

$$\mathcal{N}_K^\varepsilon(z) = \frac{1}{\varepsilon^\alpha}\int_0^z \nu_K\big(\frac{x}{\varepsilon^2}\big)dx = \int K(x)\frac{1}{\varepsilon^\alpha}\int_{-\varepsilon^2 x}^{z-\varepsilon^2 x} \nu\big(\frac{y}{\varepsilon^2}\big)dydx\,,$$

and therefore

$$\begin{aligned}\big|\mathcal{N}_K^\varepsilon(z) - \mathcal{N}^\varepsilon(z)\big| &\leq \int K(x)\frac{1}{\varepsilon^\alpha}\Big|\int_{-\varepsilon^2 x}^0 \nu\big(\frac{y}{\varepsilon^2}\big)dy - \int_{z-\varepsilon^2 x}^z \nu\big(\frac{y}{\varepsilon^2}\big)dy\Big|dx \\ &\leq 2\varepsilon^{2-\alpha}\|\nu\|_\infty \int |x|K(x)dx\,,\end{aligned}$$

which goes to zero as $\varepsilon \to 0$. ☐

**Appendix C. Proof of the Averaging Theorem.** In this Appendix we give a proof of Proposition 4.1. We fix $T > 0$ and prove the convergence in the space of continuous functions over $[-T_0, T]$ with the supremum norm $\|\cdot\|_\infty$. We first list some properties of the operators $F$ and $\tilde{F}$ defined by (4.9) and (4.11), respectively, in the following two lemmas.

Lemma C.1. *Let $A(s)$ be a deterministic continuous function. Then*

$$\mathbb{E}\left[\Big\|\frac{1}{Z}\int_0^Z [F(y)A]dy - \tilde{F}A]\Big\|_\infty\right] \overset{Z\to\infty}{\longrightarrow} 0\,.$$

*Proof.* Let us define

$$\begin{aligned}\Delta_Z(s) &:= \frac{1}{Z}\int_0^Z [F(y)A](s)dy - \tilde{F}A(s)\\ &= -\frac{\bar{c}^2}{32}\int_{-T_0}^{s}\left\{\frac{1}{Z}\int_0^Z \nu'(\bar{c}y)\nu'\Big(\bar{c}y+\bar{c}\frac{s-u}{2}\Big)\right.\\ &\qquad\qquad \left. -\mathbb{E}\left[\nu'(\bar{c}y)\nu'\Big(\bar{c}y+\bar{c}\frac{s-u}{2}\Big)\right] dy\right\} A(u)du\,.\end{aligned}$$

By (2.5) and the boundedness of the derivatives of $\nu$, we have for any $s, u \in [-T_0, T]$

$$\mathbb{E}\left[\left|\frac{1}{Z}\int_0^Z \nu'(\bar{c}y)\nu'\Big(\bar{c}y+\bar{c}\frac{s-u}{2}\Big) - \mathbb{E}\Big[\nu'(\bar{c}y)\nu'\Big(\bar{c}y+\bar{c}\frac{s-u}{2}\Big)\Big]dy\right|^2\right] \stackrel{Z\to\infty}{\longrightarrow} 0\,.$$

Therefore, by the dominated convergence theorem, for any $s \in [-T_0, T]$,

$$\mathbb{E}\left[|\Delta_Z(s)|\right] \stackrel{Z\to\infty}{\longrightarrow} 0\,.$$

It remains to control the modulus of continuity to get a uniform in $s$ estimate. From the uniform boundedness of the process $\nu'$, we have

$$\sup_{|s_1-s_2|\le\delta} |\Delta_Z(s_1) - \Delta_Z(s_2)| \le \frac{\bar{c}^2\|\nu'\|_\infty^2}{16}\sup_{|s_1-s_2|\le\delta}|A(s_1)-A(s_2)|\,.$$

Therefore, setting $s_k = -T_0 + k(T+T_0)/N$, $k = 0, \ldots, N$, we have

$$\mathbb{E}\left[\|\Delta_Z\|_\infty\right] \le \sum_{k=0}^N \mathbb{E}\left[|\Delta_Z(s_k)|\right] + \frac{\bar{c}^2\|\nu'\|_\infty^2}{16}\sup_{|s_1-s_2|\le(T+T_0)/N}|A(s_1)-A(s_2)|\,.$$

Taking first the limit $Z \to \infty$ and then $N \to \infty$ gives the result from the uniform continuity of $A$ over the compact interval $[-T_0, T]$. ◻

LEMMA C.2. *(1) For any $y$, the operators $F(y)$ and $\tilde{F}$ are uniformly Lipschitz with a nonrandom Lipschitz constant $c$:*

$$\|F(y)A - F(y)B\|_\infty \le c\|A-B\|_\infty\,, \quad \|\tilde{F}A - \tilde{F}B\|_\infty \le c\|A-B\|_\infty\,.$$

*(2) There exists $C > 0$ such that*

$$\sup_{y\in\mathbb{R}}\|F(y)A\|_\infty + \|\tilde{F}A\|_\infty \le C\|A\|_\infty\,.$$

*Proof.* The first part of the lemma follows from the uniform in $s$ estimate

$$|[F(y)A](s) - [F(y)B](s)| \le \frac{\bar{c}^2\|\nu'\|_\infty^2}{16}\|A-B\|_\infty\,,$$

which also holds true for $\tilde{F}$. The second part follows directly from the boundedness of the process $\nu'$. ◻

We can now give the proof of Proposition 4.1. It is enough to prove convergence in the mean of the supremum norm of the difference between $A^\varepsilon$ and $\tilde{A}$, because this implies convergence in probability. From the integral equation formulations

$$A^\varepsilon(s,\tau) = f(s) + \int_0^\tau F\left(\frac{y}{\varepsilon^2}\right) A^\varepsilon(s,y)dy\,, \qquad \tilde{A}(s,\tau) = f(s) + \int_0^\tau \tilde{F}\tilde{A}(s,y)dy\,,$$

the difference between $A^\varepsilon$ and $\tilde{A}$ satisfies

$$A^\varepsilon(s,\tau) - \tilde{A}(s,\tau) = \int_0^\tau \left(F\left(\frac{y}{\varepsilon^2}\right) A^\varepsilon(s,y) - F\left(\frac{y}{\varepsilon^2}\right)\tilde{A}(s,y)\right) dy + g^\varepsilon(s,\tau)\,,$$

where $g^\varepsilon(s,\tau) := \int_0^\tau F(\frac{y}{\varepsilon^2})\tilde{A}(s,y) - \tilde{F}\tilde{A}(s,y)dy$. Taking the supremum norm (in $s$), the expectation and applying Gronwall's lemma, we obtain for any arbitrary $\tau_0 > 0$,

$$\sup_{\tau\in[0,\tau_0]} \mathbb{E}\left[\|A^\varepsilon(\cdot,\tau) - \tilde{A}(\cdot,\tau)\|_\infty\right] \le e^{c\tau_0} \sup_{\tau\in[0,\tau_0]} \mathbb{E}[\|g^\varepsilon(\cdot,\tau)\|_\infty]\,.$$

It remains to show that the last term on the right goes to 0 as $\varepsilon \to 0$. Let $\delta > 0$:

$$\begin{aligned} g^\varepsilon(s,\tau) = &\sum_{k=0}^{[\tau/\delta]-1} \int_{k\delta}^{(k+1)\delta} \left(F\left(\frac{y}{\varepsilon^2}\right)\tilde{A}(s,y) - \tilde{F}\tilde{A}(s,y)\right) dy \\ &+ \int_{\delta[\tau/\delta]}^\tau \left(F\left(\frac{y}{\varepsilon^2}\right)\tilde{A}(s,y) - \tilde{F}\tilde{A}(s,y)\right) dy\,. \end{aligned}$$

Set $M_{\tau_0} = \sup_{\tau\in[0,\tau_0]} \|\tilde{A}(\cdot,\tau)\|_\infty$. From Lemma C.2, the last term of the right-hand side is bounded by $CM_{\tau_0}\delta$. Furthermore, $F$ is Lipschitz, so that

$$\left\|F\left(\frac{y}{\varepsilon^2}\right)\tilde{A}(\cdot,y) - F\left(\frac{y}{\varepsilon^2}\right)\tilde{A}(\cdot,k\delta)\right\|_\infty \le c\left\|\tilde{A}(\cdot,y) - \tilde{A}(\cdot,k\delta)\right\|_\infty \le cCM_{\tau_0}|y-k\delta|\,.$$

Similarly we have

$$\left\|\tilde{F}\tilde{A}(\cdot,y) - \tilde{F}\tilde{A}(\cdot,k\delta)\right\|_\infty \le cCM_{\tau_0}|y-k\delta|\,.$$

Therefore

$$\begin{aligned} \|g^\varepsilon(\cdot,\tau)\|_\infty \le &\left\|\sum_{k=0}^{[\tau/\delta]-1} \int_{k\delta}^{(k+1)\delta} \left(F\left(\frac{y}{\varepsilon^2}\right)\tilde{A}(\cdot,k\delta) - \tilde{F}\tilde{A}(\cdot,k\delta)\right) dy\right\|_\infty \\ &+2cCM_{\tau_0} \sum_{k=0}^{[\tau/\delta]-1} \int_{k\delta}^{(k+1)\delta} (y-k\delta)dy + 2cCM_{\tau_0}\delta \\ \le &\,\varepsilon^2 \sum_{k=0}^{[\tau/\delta]-1} \left\|\int_{k\delta/\varepsilon^2}^{(k+1)\delta/\varepsilon^2} \left(F(y)\tilde{A}(\cdot,k\delta) - \tilde{F}\tilde{A}(\cdot,k\delta)\right) dy\right\|_\infty \\ &+cCM_{\tau_0}(\tau+2)\delta\,. \end{aligned}$$

Taking the expectation and the supremum over $\tau \in [0,\tau_0]$, we get

$$\begin{aligned} &\sup_{\tau\in[0,\tau_0]} \mathbb{E}[\|g^\varepsilon(\cdot,\tau)\|_\infty] \\ &\le \delta \sum_{k=0}^{[\tau_0/\delta]-1} \mathbb{E}\left[\left\|\frac{\varepsilon^2}{\delta}\int_{k\delta/\varepsilon^2}^{(k+1)\delta/\varepsilon^2} \left(F(y)\tilde{A}(\cdot,k\delta) - \tilde{F}\tilde{A}(\cdot,k\delta)\right) dy\right\|_\infty\right] + cCM_{\tau_0}(\tau_0+2)\delta\,. \end{aligned}$$

Taking the limit $\varepsilon \to 0$, we obtain from Lemma C.1

$$\limsup_{\varepsilon\to 0} \sup_{\tau\in[0,\tau_0]} \mathbb{E}[\|g^\varepsilon(\cdot,\tau)\|_\infty] \leq cCM_{\tau_0}(\tau_0+2)\delta\,.$$

Finally, letting $\delta \to 0$ completes the proof of Proposition 4.1.

REFERENCES

[1] M. Asch, W. Kohler, G. Papanicolaou, M. Postel, and B. White, Frequency content of randomly scattered signals, SIAM Review, **33**, (1991), pp. 519-626.

[2] J.-M. Bardet, G. Lang, G. Oppenheim, A. Philippe, and M. S. Taqqu, Generators of the long-range dependence processes: a survey, in Theory and applications of long-range dependence, P. Doukhan, G. Oppenheim, and M. S. Taqqu, eds., Birkhauser (2003), pp. 579-624.

[3] L. Berlyand and R. Burridge, The accuracy of the O'Doherty-Anstey approximation for wave propagating in highly disordered stratified media, Wave Motion, **21**, (1995), pp. 357-373.

[4] D. T. Blackstock, Generalized Burgers equations for plane waves, J. Acoust. Soc. Am., **77**, (1985), pp. 2050-2053.

[5] R. Burridge and H. W. Chang, Multimode one-dimensional wave propagation in a highly discontinuous medium, Wave Motion, **11**, (1989), pp. 231-249.

[6] R. Burridge, G. Papanicolaou, and B. White, One-dimensional wave propagation in a highly discontinuous medium, Wave Motion, **10**, (1988), pp. 19-44.

[7] M. Caputo, Linear models of dissipation whose $Q$ is almost frequency independent II, Geophys. J. R. Astron. Soc., **1**, (1967), pp. 529-539.

[8] M. Caputo and F. Mainardi, A new dissipation model based on memory mechanism, Pure Appl. Geophys., **91**, (1971), pp. 134-147.

[9] W. Chen and S. Holm, Fractional Laplacian time-space models for linear and nonlinear lossy media exhibiting arbitrary frequency power-law depency, J. Acoust. Soc. Am., **115**, (2003), pp. 1424-1430.

[10] J.-F. Clouet and J.-P. Fouque, Spreading of a pulse traveling in random media, Ann. Appl. Probab., **4**, (1994), pp. 1083-1097.

[11] S. Dolan, C. Bean, and B. Riollet, The broad-band fractal nature of heterogeneity in the upper crust from petrophysical logs, Geophys. J. Int., **132**, (1998), pp. 489-507.

[12] P. Doukhan, Models, inequalities, and limit theorems for stationary sequences, in Theory and applications of long-range dependence, P. Doukhan, G. Oppenheim, and M. S. Taqqu, eds., Birkhauser (2003), pp. 39-100.

[13] A. Fannjiang and K. Sølna, Scaling limits for wave beams in atmospheric turbulence, Stoch. Dyn., **4**, (2004), pp. 135-151.

[14] W. Feller, An introduction to probability theory and its applications, Vol. 2, Wiley, New York, 1971.

[15] J.-P. Fouque, J. Garnier, G. Papanicolaou, and K. Sølna, Wave propagation and time reversal in randomly layered media, Springer, New York, 2007.

[16] A. E. Gargett, The scaling of turbulence in the presence of stable stratification, J. Geophys. Res., **93**, (1988), pp. 5021-5036.

[17] S. Gelinsky, S. A. Shapiro, and T. Müller, Dynamic poroelasticity of thinly layered structure, Int. J. Solids Structures, **35**, (1998), pp. 4739-4751.

[18] A. Hanyga and V. E. Rok, Wave propagation in micro-heterogeneous porous media: A model based on an integro-differential wave equation, J. Acoust. Soc. Am., **107**, (2000), pp. 2965-2972.

[19] P. Lewicki, Long-time evolution of wavefronts in random media, SIAM J. Appl. Math., **54**, (1994), pp. 907-934.

[20] P. Lewicki, R. Burridge, and G. Papanicolaou, Pulse stabilization in a strongly heterogeneous medium, Wave Motion, **20**, (1994), pp. 177-195.

[21] R. Marty, Asymptotic behavior of differential equations driven by periodic and random processes with slowly decaying correlations, ESAIM: Probability and Statistics, **9**, (2005), pp. 165-184.

[22] R. Marty and K. Sølna, Acoustic waves in long-range random media, accepted in SIAM J. Appl. Math., available at http://www.math.uci.edu/~ksolna.

[23] A. Nachbin and K. Sølna, Apparent diffusion due to topographic microstructure in shallow waters, Phys. Fluids, **15**, (2003), pp. 66-77.

[24] R. F. O'Doherty and N. A. Anstey, Reflections on amplitudes, Geophysical Prospecting, **19**, (1971), pp. 430-458.

[25] C. Sidi and F. Dalaudier, Turbulence in the stratified atmosphere: Recent theoretical developments and experimental results, Adv. in Space Res., **10**, (1990), pp. 25-36.

[26] K. Sølna, Acoustic pulse spreading in a random fractal, SIAM J. Appl. Math., **63**, (2003), pp. 1764-1788.

[27] K. Sølna and G. C. Papanicolaou, Ray theory for a locally layered medium, Waves in Random Media, **10**, (2000), pp. 151-198.

[28] N. V. Sushilov and R. S. C. Cobbold, Frequency domain wave equation and its time-domain solutions in attenuating media, J. Acoust. Soc. Am., **115**, (2003), pp. 1431-1435.

[29] T. L. Szabo, Time domain wave equations for lossy media obeying a frequency power law, J. Acoust. Soc. Am., **96**, (1994), pp. 491-500.

[30] M. Taqqu, Convergence of integrated processes of arbitrary Hermite rank, Z. Wahrscheinlichkeitstheorie verw. Gebiete, **50**, (1979), pp.53-83.